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Children's Acquisition of Arithmetic Principles: The Role of Experience

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The current study investigated how young learners' experiences with arithmetic equations can lead to learning of an arithmetic principle. The focus was elementary school children's acquisition of the Relation to Operands principle for subtraction (i.e., for natural numbers, the difference must be less than the minuend). In Experiment 1, children who viewed incorrect, principle-consistent equations and those who viewed a mix of incorrect, principle-consistent and principle-violation equations both showed gains in principle knowledge. However, children who viewed only principle-consistent equations did not. We hypothesized that improvements were due in part to improved encoding of relative magnitudes. In Experiment 2, children who practiced comparing numerical magnitudes increased their knowledge of the principle. Thus, experience that highlights the encoding of relative magnitude facilitates principle learning. This work shows that exposure to certain types of arithmetic equations can facilitate the learning of arithmetic principles, a fundamental aspect of early mathematical development.

How do people acquire knowledge of general principles in domains such as mathematics, language, and science? What role does experience play in the acquisition of principle knowledge? In the case of mathematics, learners may acquire principle knowledge from experience with many types of examples, including both correct and incorrect examples. This article presents a pair of experiments that investigate how experience facilitates learning of principles in arithmetic. To address this issue, we draw on what is known about

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children's arithmetic principle knowledge and their learning of regularities in other domains. The results can inform both developmental theory and educational practice.

PRINCIPLE LEARNING

Principles can be defined as general rules or regularities that correspond to concepts within a domain. For example, in arithmetic, when adding natural numbers, the result is always bigger than either addend. Arithmetic principles are an important aspect of mathematical knowledge (Dixon & Bangert, 2005; Canobi, 2005; Prather & Alibali, 2009; Rasmussen, Ho, & Bisanz, 2003). However, principle learning is also important in many other domains, including counting (e.g., Gelman & Gallistel, 1978), language acquisition (e.g., Aslin, Saffran, & Newport, 1998), proportional reasoning (e.g., Dixon & Moore, 1996), and physics (e.g., Chi, Feltovich, & Glaser, 1982). Principle knowledge is integral to having generalizable and flexible knowledge of a problem domain.

The current study is concerned with the *Relation to Operands* principle. *Relation to Operands* describes the relationships between the operands in a given arithmetic equation and the result of the operation. The exact relationship varies depending on the specific operation. For natural numbers, in simple addition equations ($a+b=c$), the sum (c) must be greater than the two addends (a and b). In simple subtraction equations ($a-b=c$), the difference (c) must be less than the minuend (a); however, it may have any relationship with the subtrahend (b). For example, for the equation $5+3=[?]$, the solution *4* is a violation of the Relation to Operands principle, whereas the solutions *8* and *12* are not.

We focus on this principle because it is fundamental to an understanding of arithmetic operations, and because it is of particular importance in early mathematical development. The principle captures fundamental properties of arithmetic operations. As such, it is one aspect of people's conceptual understanding of arithmetic operations, or their "operation sense" (Slavit, 1998).

Our specific focus is on the Relation to Operands principle for subtraction in symbolic format. Past research suggests that knowledge of subtraction concepts is quite fragile in the early elementary years (Baroody, 1999; Baroody, Lai, Li, & Baroody, 2009) and that children tend to learn subtractive relations later than they learn additive relations (Canobi, 2005). The inverse relationship between addition and subtraction in symbolic contexts is not well understood by many early elementary school children (Baroody, 1999; Canobi, 2005). Further, previous research suggests that children's knowledge of the Relation to Operands principle for subtraction is still developing during elementary school (Dixon, Deets, & Bangert, 2001; Prather & Alibali, 2009). Dixon et al. (2001) suggest that this is due to children's less extensive experience with subtraction, relative to addition.

Only fairly recently have researchers begun to directly investigate the learning mechanisms involved in acquiring arithmetic principles (Canobi, 2009; Dixon & Bangert, 2005; Lai, Baroody, & Johnson, 2008; Prather & Alibali, 2008a; Siegler & Stern, 1998). The general paradigm used in most of these studies is to investigate how principles are learned in the course of solving arithmetic equations for which the principle is relevant. In contrast, in this research, we investigate what students learn through exposure to examples. Learners are exposed to many examples in the course of their mathematical development. Further, learners can view examples of arithmetic equations before they are facile in actually solving the equations. Thus, learning from exposure to arithmetic examples may be particularly applicable in early development.

We hypothesize that learners may acquire principle knowledge through exposure to structured examples. Learning of regularities through exposure to examples is a form of implicit learning. The literature on implicit learning has focused largely on two types of tasks: complex control systems and artificial grammars. In studies of learning about complex control systems (e.g., Berry & Broadbent, 1984; Broadbent, FitzGerald, & Broadbent, 1986), participants practice operating on a system of variables to control a single dependent outcome variable. Participants eventually gain a high degree of skill; however, they remain very poor at explaining which variables affect the outcome. In studies of artificial grammar learning (e.g., Gomez & Schvaneveldt, 1994; Knowlton & Squire, 1996; Seger, 1994; Tunney & Altmann, 1999), participants are exposed to examples that are consistent with a predetermined finite-state grammar. The examples are typically strings of letters of varying length (such as ABFE or ABBQW). Participants may be instructed to memorize the examples or otherwise pay attention to them, and they are not told that there are any regularities in the stimuli to learn. After the initial exposure, participants' knowledge of the regularity is assessed via their evaluation of novel examples that either violate or are consistent with the regularity. The general consensus is that participants are often able to learn what seem like complex regularities through exposure to consistent examples. In both complex control systems and artificial grammar learning, participants' behavior is dependent on aspects of the environment that are learned through experience with the stimuli—either simple exposure, in the case of artificial grammar learning, or experience with the causal relations involved in the system, in the case of complex control systems.

In most implicit learning studies, all of the examples that participants encounter are consistent with the principle. However, a body of research on analogical reasoning (Gentner & Medina, 1998), inductive reasoning (Kalish & Lawson, 2007), categorization (Namy & Clepper, 2010), contrasting cases (Rittle-Johnson & Star, 2007), and cognitive conflict (e.g., Eryilmaz, 2002; Groβe & Renkl, 2007) suggests that exposure to both principle-consistent and principle-violation examples may facilitate learning more than exposure to principle-consistent, error-free examples alone. In one such study, elementary school students who explained both correct and incorrect solutions to mathematical equivalence problems (e.g., $3+4+6=\_\_+6$) learned more than students who explained only correct solutions (Siegler, 2002). In another study, learners' knowledge of decimal concepts improved when they spent time considering conceptually incorrect examples in addition to correct examples (Huang, Liu, & Shiu, 2008). For example, a student may be asked "Does the 4 in 5.4 mean that there are 'four' pancakes?" This question poses an incorrect interpretation, and should be answered with "no."

In general, learners who are exposed to both correct and incorrect examples have access to more information about regularities in the domain. In support of this idea, contrasting different types of examples is a common practice in mathematics instruction, particularly in high-performing countries (Richland, Zur, & Holyoak, 2007; Stigler & Hiebert, 1999). Thus, although many implicit learning tasks show learning and generalization from principle-consistent examples only, other evidence suggests that learners stand to benefit when contrasts between principle-consistent and principle-violation examples are made available. We hypothesize that this is the case for arithmetic principles.

ARITHMETIC PRINCIPLE KNOWLEDGE DEPENDS ON ENCODING OF EQUATIONS

Changes in learners' knowledge have been attributed to changes in their encoding in the problem domain (Karmiloff-Smith, 1992; McNeil & Alibali, 2004, 2005; Siegler, 1976). Encoding can be defined as the uptake of information from the environment into working memory. For any given stimulus, whether it is a simple arithmetic equation, a chess board scene, or a physics problem, there are many features that can be encoded by the learner. For example, consider the simple equation $8+3=11$. Learners could note the color of the numerals, the order of the numbers, the type of operation, the value of the operands, or the relative magnitude of the operands and the sum. The learner uses these encoded features to form an internal representation of the equation.

This internal representation may also draw on other sources of knowledge from long-term memory, such as prior knowledge of common problem schemas.

Problem encoding and representation vary as a function of domain knowledge or expertise. Studies comparing novices and experts in several domains (e.g., chess, physics) have shown that experts more accurately encode relevant displays (Chase & Simon, 1973; Larkin, McDermott, Simon, & Simon, 1980). Chess experts do not generally have better memory than chess novices; however, they are superior in encoding the positions of chess pieces in actual game positions. In the domain of arithmetic, deficits in encoding equations have been linked to difficulties in solving similar equations (McNeil & Alibali, 2004).

The current study addresses encoding as it relates to arithmetic principle knowledge. Problem encoding may change over the course of development (Karmiloff-Smith, 1992), with the development of expertise in a domain (Chase & Simon, 1973), or in the short term given particular sorts of experience (Alibali, Phillips, & Fischer, 2009; Siegler & Stern, 1998). We hypothesize that changes in encoding are one mechanism that leads to gains in principle knowledge.

OVERVIEW OF EXPERIMENTS

In the experiments that follow, we examine whether changes in encoding and exposure to different types of examples contribute to the acquisition of principle knowledge. In Experiment 1, we test the possibility that learners may acquire the Relation to Operands principle for subtraction through exposure to example equations with different types of errors, as in prior work on implicit learning and cognitive conflict. Specifically, we compare learning from exposure to examples that are consistent with the principle and examples that violate the principle. For example, for the correct equation $12 - 3 = 9$, one could also consider an incorrect principle-consistent equation, such as $12 - 3 = 4$, or an incorrect principle-inconsistent equation (i.e., a principle violation), such as $12 - 3 = 14$. Incorrect principle-consistent equations still hold the general relational structure that defines the Relation to Operands principle, whereas incorrect principle-inconsistent equations do not.

We suggest that in considering examples with errors, participants may note the importance of the relative magnitudes of the numbers involved in subtraction equations, and they may adjust their encoding of subtraction equations to emphasize relative magnitude. They may ultimately draw on this improved encoding of relative magnitude to infer the regularity that in subtraction equations (with natural numbers), the result is always less than the starting number. In Experiment 2, we test the role of encoding in principle learning using a more direct manipulation of encoding, by providing participants with practice attending to relative magnitudes.

EXPERIMENT 1

This experiment tests the possibility that learners may acquire the Relation to Operands principle for subtraction through exposure to examples. We hypothesize that experience with principle-consistent incorrect and principle-inconsistent examples will highlight the relevant regularities for learners and subsequently lead to changes in learners' encoding and increased principle knowledge. Contrasting different types of examples may promote improved encoding of problem features relevant for the principle. We hypothesized that this would be most likely in the condition that involved principle-inconsistent equations, incorrect principle-consistent equations, and correct equations, as opposed to only correct and incorrect principle-consistent equations, or only correct equations. This is because of the contrast in the specific feature relevant to Relation to Operands—the magnitude relations obtained between the operands and the result. This contrast should make this feature more salient to the learner and hence more likely to be encoded.

Method

Participants

Participants were children in Grades 2 ($n = 81$), 3 ($n = 91$), and 4 ($n = 104$). This range of grade levels was selected because prior work suggested that children at these grade levels may not have knowledge of the Relation to Operands principle for subtraction (Prather & Alibali, 2007). We did not collect information about children's birth dates; however, in Grades 2 through 4, children range from 7 to 10 years old. Participants were recruited through local parochial schools, and testing was conducted in students' classrooms. A small subset of participants ($n = 14$, including 6 second-grade students, 1 third-grade student, and 7 fourth-grade students, distributed roughly equally across conditions) did not complete the worksheet; their data were not included in the analysis.

All of the children were familiar with subtraction with natural numbers. We reviewed the math textbooks used by the students and found that none of the texts explicitly addressed Relation to Operands.

Procedure

The study was conducted in participants' classrooms and took approximately 20 minutes, not including instructions and setup. Worksheet booklets were passed out, and instructions were given orally to each class as a whole. Each worksheet booklet consisted of a pretest principle knowledge assessment, a training task, an equation-encoding task, and a posttest principle knowledge assessment.

Pretest principle knowledge assessment. Participants engaged in an equation evaluation task that was used to assess their principle knowledge. Participants were asked to evaluate examples that did or did not violate the target principle. Similar tasks that require participants to evaluate both principle violations and incorrect nonviolations have been used in many prior studies of arithmetic principle knowledge (e.g., Dixon & Bangert, 2005; Dixon et al., 2001; Prather & Alibali, 2008a, 2008b; see Prather & Alibali, 2009, for review). This task was administered as part of the worksheet booklet.

In the equation evaluation task, participants were instructed to look over sets of equations that had been produced by pairs of fictional students and decide if one student understood arithmetic better or if the two students understood the same amount. Thus, for each pair of students, participants selected one of three options: one or the other of the two fictional students understood math better, or the two students understood the same amount. The task was introduced with the following instructions:

> The worksheet that I passed out has some math problems that have been solved by some other students. I'd like you to tell me what you think by answering the questions. The front page has an example. Trevor and Luke both solved five equations. The ones they got wrong are marked with an X. They both got three wrong. I took a look at what their answers were for each equation and decided that I thought that Luke understood math better and circled his name.
>
> In this worksheet, there are more students' equations. I want you guys to do the same. Take a look at the equations the students solved and decide who understands math better. The ones they don't get right are marked with an X. Make sure to circle your answer. When you are done, I will walk around and collect your worksheets.

The two fictional students in each pair solved the same five equations. For each equation, either both students in the pair provided the correct solution or both solved the equation incorrectly. For the incorrect equations (three of the five equations), one student produced all principle-violation solutions (i.e., principle-inconsistent incorrect solutions) while the other produced all principle-consistent incorrect solutions (see the Appendix for examples). The deviation from the correct answer was the same across the two students in each pair, and the two students' work was displayed on the same page. This task gave participants an opportunity to indicate that producing incorrect principle-consistent equations was better than producing incorrect principle-violation equations. There were four items (i.e., four pairs of students) on the pretraining principle knowledge assessment, so participants had four opportunities to display this knowledge.

Training task. On two pages, participants were presented with four columns of solved subtraction equations, each solved by one of two hypothetical students. The participants were instructed to read through and mark the incorrect equations. Participants viewed 40 equations in total, in sets of 10. Participants were assigned to one of three conditions: 1) *correct*: correct equations only; 2) *principle-consistent*: a mixture of correct and incorrect principle-consistent equations (2 correct and 8 incorrect principle-consistent equations in each set of 10); or 3) *principle-inconsistent*: a mixture of correct, incorrect principle-consistent and incorrect principle-violation equations (2 correct and 4 incorrect principle-consistent, and 4 incorrect principle-violation equations in each set of 10). This task was administered to students in the classroom setting as part of the worksheet booklet. Students within each participating classroom were randomly assigned to conditions. Across conditions and grade levels, participants' performance on the training task averaged 95%.

By providing the stimuli all at once as opposed to serially, the memory demands of the task were minimized. If learning occurs the way we theorize, the learner must note general patterns across many equations. A display in which the learner views many equations at once under no time constraints may be most effective for young learners.

Equation-encoding task. On one page of the worksheet booklet, participants viewed a circled arithmetic equation. Participants were instructed to indicate whether the equation was correct or incorrect. After turning in their booklet, participants received a second sheet with a "bonus question" that asked them to recall the circled equation and to indicate whether the first number in the circled equation was larger or smaller than the result. Participants who correctly remembered the relative magnitudes of the numbers were inferred to have encoded the relative magnitudes correctly. Participants' booklets were collected before the "bonus question" so that they could not look back at the original equation.

Posttest principle knowledge assessment. The posttest principle knowledge assessment used the same method, though not the exact same stimuli, as the pretest principle knowledge assessment. Cronbach's alpha, a common measure of reliability, for the principle knowledge measure was .73 at pretest and .78 at posttest, both of which exceed the commonly recommended minimum acceptable level of .6 (Hair, Anderson, Tatham, & Black, 1998).

Results

What Did Students in Each Grade Know Before Training?

We first examined participants' pretest knowledge of the Relation to Operands principle for subtraction as a function of grade. On the pretest and posttest principle knowledge assessments, points were awarded for each comparison in which the participant indicated that the fictional student who did not violate the principle understood arithmetic better than the student who violated the principle. There were four items on each test; thus, each participant received a score from 0 to 4 at pretest and at posttest. This task involved a forced multiple choice between three options; thus, chance would yield an average total score of 1.33.

There was a main effect of grade on participants' pretest scores, $F(2, 273) = 10.64$, $p < .01$, $\eta^2 = .07$. Second-grade students ($n = 81$) averaged 1.59 points ($SD = 1.46$), third-grade students ($n = 91$) averaged 1.85 points ($SD = 1.48$), and fourth-grade students ($n = 104$) averaged 2.54 points ($SD = 1.44$), as seen in Figure 1. The second graders' performance did not

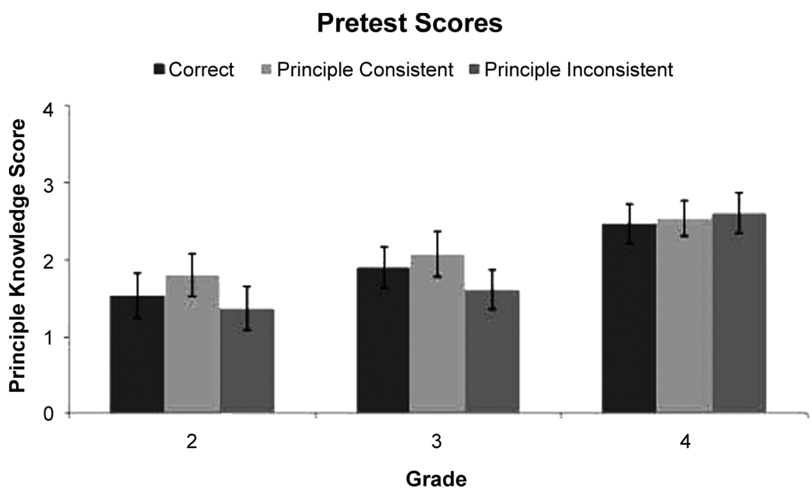

FIGURE 1 Pretest principle knowledge scores for participants in grades 2, 3, and 4 by experimental condition. The error bars represent standard errors.

differ from chance, $t(80) = 1.61$, $p = .11$, $d = 0.17$. Third graders performed above chance, $t(90) = 3.32$, $p = .001$, $d = 0.34$, and so did fourth graders, $t(104) = 7.94$, $p < .001$, $d = 0.83$.

Did Participants' Principle Knowledge Scores Increase After the Training Task? Did Gains Depend on Which Type of Training Stimuli They Viewed?

We first examined effects of training for the full sample of participants. A mixed-model 2 (test: pretest or posttest) $\times$ 3 (condition) $\times$ 3 (grade) analysis of variance (ANOVA) yielded a significant main effect of grade, $F(2, 267) = 13.6$, $p <. 01$, $\eta_p^2 = .09$, but no main effects of test, $F(1, 267) = 1.85$, $p = .17$, $\eta_p^2 = .007$, or condition $F(2, 267) = 1.51$, $p = .21$, $\eta_p^2 = .011$. Interactions between the factors were also nonsignificant: test $\times$ condition, $F(2, 267) = 0.6$, $p = .55$, $\eta_p^2 = .004$; test $\times$ grade, $F(2, 267) = 0.57$, $p = .59$, $\eta_p^2 = .004$; condition $\times$ grade, $F(4, 267) = 0.87$, $p = .48$, $\eta_p^2 = .013$; test $\times$ grade $\times$ condition, $F(4, 267) = 1.32$, $p = .24$, $\eta_p^2 = .02$).

Because this experiment focused on the acquisition of principle knowledge, we also analyzed the subsample of participants who began with low principle knowledge scores on the pretest. For this analysis, we included participants who scored below 2 on the pretest ($n = 109$), for two reasons. First, the median pretest score was 2. Second, given that there were three response options and four items, chance performance alone would yield a score of 1.33.

For this subsample, a mixed-model 2 (test: pretest or posttest) $\times$ 3 (condition) $\times$ 3 (grade) ANOVA yielded a significant overall effect of test, $F(1, 100) = 16.23$, $p < .01$, $\eta_p^2 = .12$, and a significant interaction between test and condition, $F(2, 100) = 2.94$, $p = .05$, $\eta_p^2 = .055$. The main effect of condition was not significant, $F(2, 100) = 0.50$, $p = .60$, $\eta_p^2 = .01$, nor was the main effect of grade, $F(2, 100) = 1.23$, $p = .29$ $\eta_p^2 = .024$, the test $\times$ grade interaction, $F(2, 100) = 0.186$, $p = .83$, $\eta_p^2 = .004$, the condition $\times$ grade interaction, $F(4, 100) = 1.33$, $p = .26$, $\eta_p^2 = .051$, or the test $\times$ grade $\times$ condition interaction, $F(4, 100) = 0.77$, $p = .54$, $\eta_p^2 = .03$.

Planned contrasts that examined change in scores from pretest to posttest indicated that participants in both the principle-inconsistent ($M = 0.57$, $SD = 1.06$) and principle-consistent ($M = 0.55$, $SD = 1.08$) conditions improved significantly more than participants in the correct condition ($M = 0.05$, $SD = 0.69$), $t(106) = 2.29$, $p = .024$, $d = 0.57$, and $t(106) = 2.27$, $p = .025$, $d = 0.53$, respectively (see Figure 2). Improvement in the principle-inconsistent and principle-consistent conditions did not differ, $t(106) = 0.09$, $p = .93$, $d = 0.17$.

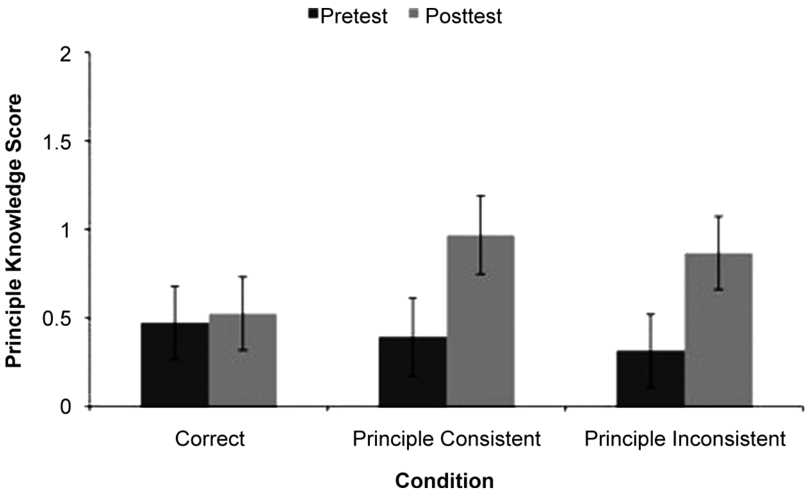

FIGURE 2 Pretest to posttest principle knowledge scores by condition, for participants who scored <2 on the pretest principle knowledge assessment. The error bars represent standard errors.

Were Learners' Equation Encoding and Principle Knowledge Related?

This experiment included a measure of equation encoding, which was based on a single problem. This measure was administered at the end of the study, and it referred to an equation that was presented after the training task. Thus, we could not examine change in encoding as a function of training, but we could examine the relation between encoding and principle knowledge at posttest.

Overall, 60% of participants answered the encoding problem correctly. This included 53% of second graders, 65% of third graders, and 60% of fourth graders. Broken down by condition, this included 68% of participants in the correct-only condition, 56% of participants in the principle-consistent condition, and 56% of participants in the principle-inconsistent condition. We hypothesized that learners who accurately encoded magnitude relations (i.e., who correctly answered whether the first number was smaller or bigger than the last) would display more principle knowledge than those who did not encode magnitude relations. Considering the sample as a whole, the means were in the predicted direction; however, the difference in posttest performance between participants who correctly encoded magnitude relations in the target equation ($M = 1.99$, $SD = 1.62$) and those who did not ($M = 1.83$, $SD = 1.49$) was small and not significant, $t(274) = 0.81$, $p = .41$, $d = 0.10$. There was also no significant association between condition and encoding performance, $\chi^2(2, N = 274) = 3.75$, $p = .15$.

Experiment 1 Discussion

This study demonstrated that exposure to equations can affect learners' acquisition of the *Relation to Operands* principle. Children showed differential gains in principle knowledge as a function of their exposure to different types of arithmetic equations. In contrast to work on principle learning in other domains (e.g., artificial grammar learning), participants did not benefit when they were exposed only to correct examples. Put another way, participants who viewed solely correct equations showed no change in principle knowledge during the experiment. This result is consistent with work on cognitive conflict (e.g., Huang et al., 2008), which suggests that experience with conflicting examples (rather than solely correct examples) is especially likely to provoke changes in principle knowledge. This is an important finding, as formal education often involves maximizing experience with correct equations (Stigler & Hiebert, 1999).

We hypothesized that exposure to contrasting examples would promote improved encoding of problem features that vary across examples. When examples vary in terms of features relevant to the principle, the contrast should make those features more salient, hence more likely to be encoded, and ultimately more likely to be used in building principle knowledge. Thus, we predicted that participants would show the greatest gains in principle knowledge in the condition that included both principle-consistent and principle-violation examples, because these examples contrast in the specific feature relevant to Relation to Operands—the magnitude relations that are obtained between the operands and the result. Indeed, participants did benefit when they had the opportunity to compare principle-consistent and principle-violation examples. This finding is consistent with prior work on arithmetic principle acquisition in adults, which suggested that training sets that include principle-violation equations lead to substantial gains in principle knowledge (Prather & Alibali, 2008a).

However, in the present study, we also found that participants benefited to the same extent in the condition that included only principle-consistent and correct examples (and no principle violations). Put another way, the training set that included principle-consistent incorrect equations but no principle violations was as effective at promoting learning as the training set that included principle violations. Why might this be the case? It may be that the crucial variable is getting the learner to contrast differing solutions to arithmetic equations. These findings echo those from other studies suggesting that comparison is an effective means of promoting mathematics learning (e.g., Rittle-Johnson & Star, 2007). Comparison may promote deeper processing of the material, which in turn may make learners more likely to notice a regularity, or which may promote more accurate encoding of problem features.

Of course, there are limits to the conclusions that can be made based on this experiment. The three training conditions we used did not exhaust the possible types of experience that may facilitate learners' principle acquisition. It is possible that viewing principle violations alone might also lead to increased principle knowledge. It is also plausible that there may be a threshold effect, such that children need to see only a few examples of principle-consistent errors to benefit. Future studies will be needed to test these possibilities.

The results suggest that an approach to arithmetic principle learning that includes exposure to different types of examples can be quite effective. Though participants in this experiment showed learning based only on a small amount of exposure, it is likely that repeated exposure would be necessary in educational practice. And of course, we do not recommend that simple exposure to examples should take the place of other types of instruction. Instead, we suggest that implicit learning of arithmetic principles may be used to supplement other types of instruction, including more explicit or direct instruction. Exposure to varying examples may be particularly useful because it may be introduced to learners early, before they are skilled enough to solve arithmetic equations on their own.

EXPERIMENT 2

Experiment 1 showed that exposure to diverse example problems led to gains in principle knowledge. We hypothesize that experience with different types of equations may lead to improvements in problem encoding, which in turn informs principle knowledge. Although the encoding results in Experiment 1 were not as expected, we believe that improvements in problem encoding may be one mechanism underlying the effects observed in Experiment 1. To address this possibility, we more directly examine the possible link between improvements in problem encoding and gains in principle knowledge in Experiment 2.

In this experiment, we sought to directly address the question of whether changes in problem encoding can lead to gains in principle knowledge. We did this by manipulating participants' encoding. Specifically, this experiment investigated whether giving children practice in encoding relative magnitude would lead to learning of the Relation to Operands principle for subtraction.

Method

Participants

Participants ($N = 107$) were second-grade students recruited through a local parochial school. Testing was conducted in their classrooms. We chose to work with second graders because children at this grade level, as a group, showed chance performance on the principle knowledge pretest in Experiment 1. We did not collect information about children's birth dates; however, in Grade 2, children range from 7 to 8 years old. A small subset of participants ($n = 16$) were not included in the analyses due to experimenter error ($n = 7$) or not completing the worksheet ($n = 9$).

All of the children were familiar with subtraction with natural numbers. We reviewed the math textbook used by the students and found that it did not explicitly address Relation to Operands.

Procedure

The procedure was similar to that used in Experiment 1. Participants were given worksheets that took approximately 20 minutes to complete, not including instruction and setup.

Pretest principle knowledge assessment. The pretest principle knowledge assessment was identical to that used in Experiment 1.

Training task. On two pages, participants were shown rows of three numbers each. At the top of each page, participants were instructed to circle certain types of numbers. Participants in the *relation* condition were asked to circle the biggest (or smallest) number in each row. This condition was expected to foster children's encoding of the relative magnitudes of numbers, which is crucial to the Relation to Operands principle. Participants in the *parity* condition were asked to circle the odd (or even) number in each row. This condition was expected to foster participants' encoding of parity, which is not relevant to the Relation to Operands principle. In the parity condition, participants averaged 85% correct on the training task, and in the relation condition, they averaged 82% correct.

Equation-encoding task. The equation-encoding task was identical to that used in Experiment 1.

Posttest principle knowledge assessment. The posttest principle knowledge assessment used the same method as the pretest assessment.

Results

What Did Students Know at Pretest?

Scoring of the pretest and posttest principle knowledge assessments was identical to Experiment 1; each participant received a score from 0 to 4 at pretest and at posttest. As in Experiment 1, chance would yield an average total score of 1.33. Overall, participants' scores on the pretest were statistically above chance ($M = 1.72$, $SD = 1.35$), $t(89) = 2.81$, $p < .01$, $d = 0.29$, suggesting some knowledge of the principle prior to training in the overall sample, unlike the second-grade students in Experiment 1. This finding suggests that a younger age group may have been warranted for this experiment; however, our pilot testing indicated that the task was too difficult for most first graders, so we limited the sample to second graders.

Did Participants' Principle Knowledge Scores Increase After the Encoding Manipulation? Did Improvement Depend on Which Type of Encoding Practice They Received?

We first examined effects of the manipulation in the full sample of participants. A mixed-model 2 (test: pretest or posttest) $\times$ 2 (condition) ANOVA yielded a main effect of test, $F(1, 89) = 8.35$, $p = .005$, $\eta_p^2 = .086$, but no effect of condition, $F(1, 89) = 0.04$, $p = .84$, $\eta_p^2 < .0001$, and no test $\times$ condition interaction, $F(1, 89) = 0.58$, $p = .44$, $\eta_p^2 = .006$.

Because this experiment focused on the acquisition of principle knowledge, we also conducted an analysis on the subsample of participants who had low principle knowledge scores on the pretest (i.e., scores below chance, <2 on the pretest principle knowledge assessment, $n = 43$). We hypothesized

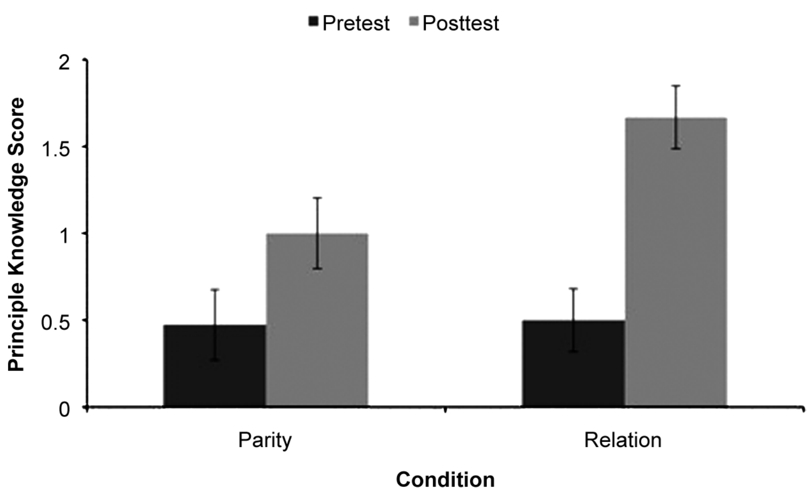

FIGURE 3 Pretest and posttest principle knowledge scores by condition, for participants who scored <2 on the pretest principle knowledge assessment. The error bars represent standard errors.

that participants in the relation condition, who practiced encoding relative magnitude during the training session, would show greater gains in principle knowledge than participants in the parity condition, who practiced encoding the parity of the numbers. To test this hypothesis, we conducted a mixed-model 2 (test: pretest or posttest) $\times$ 2 (condition) ANOVA. This analysis yielded a significant effect of test, $F(1, 41) = 19.45$, $p < .01$, $\eta_p^2 = .32$, but no main effect of condition, $F(1, 41) = 2.54$, $p = .15$, $\eta_p^2 = .048$. The test $\times$ condition interaction also did not reach significance, $F(1, 41) = 2.78$, $p = .10$, $\eta_p^2 = .064$ (see Figure 3). There were no differences between conditions at pretest, $F(1, 41) = 0.009$, $p = .92$, $\eta_p^2 = .001$. However, planned contrasts that examined change in scores from pretest to posttest revealed that participants in the relation condition ($M = 1.17$, $SD = 1.30$) improved more than participants in the parity condition ($M = 0.52$, $SD = 1.17$), $F(1, 41) = 6.03$, $p = .02$, $\eta_p^2 = .10$; see Figure 3).

Were Learners' Equation Encoding and Principle Knowledge Related?

Because there was only one encoding assessment, administered at posttest, we were not able to examine change in encoding as a function of encoding practice. However, we could examine the relation between encoding and principle knowledge at posttest.

Overall, 46% of participants in the relation condition and 42% of participants in the parity condition succeeded on the encoding item. Of course, these values are difficult to interpret because we do not have corresponding pretest data. We hypothesized that learners who encoded magnitude relations in equations would display more principle knowledge at posttest than those who did not, regardless of condition. As predicted, participants who succeeded on the encoding item after training scored higher on the posttest than those who did not ($M = 2.43$, $SD = 1.56$ vs. $M = 1.81$, $SD = 1.41$), $t(90) = 1.85$, $p = .03$, one-tailed, $d = 0.41$.

Experiment 2 Discussion

This experiment examined the effect of practice in encoding the relative magnitudes of numbers on children's understanding of the Relation to Operands principle. For participants who began with low knowledge of the Relation to Operands principle, practice encoding relative magnitudes led to gains in principle knowledge, more so than practice encoding parity. This suggests that encoding of relative magnitude is an important part of acquiring knowledge of the Relation to Operands principle.

One might question whether participants in the relation condition were actually learning the principle or just learning to notice a relation that they already understood. In our view, their improved performance on the principle knowledge task at posttest suggests that participants did indeed learn something about the principle. In this regard, it seems likely that encoding relative magnitudes is necessary but not sufficient for acquisition of the Relation to Operands principle. Learners must not only encode relative magnitudes in the arithmetic equation, but they must also note which pattern of magnitudes is consistent with subtraction and which pattern is not. Participants could have begun to notice relative magnitudes but then used that encoded feature incorrectly (i.e., by choosing the student who produced solutions that were *larger* than the starting number as having better knowledge). Accurate encoding of a regularity is a first step in constructing principle knowledge, but it is not the end of the line: Participants need to build on this knowledge by drawing out the implications of that regularity—in this case, by noting which pattern of relative magnitudes is consistent with subtraction and which pattern is not. Given that participants who received practice encoding relative magnitudes increased their principle knowledge, we infer that some participants were able to do just that.

GENERAL DISCUSSION

Empirical Summary

In Experiment 1, we showed that learners who had the opportunity to compare correct and incorrect problem examples showed greater gains in principle knowledge than learners who were exposed to correct examples only. In Experiment 2, we showed that learners who practiced encoding a problem feature that was relevant to the principle also showed greater gains in principle knowledge. These findings suggest two main conclusions. First, the types of examples learners view, even if they are not working through those examples, can influence their learning. Second, changes in how learners encode examples can also influence learning. We suggest that changes in encoding of relevant problem features may be one mechanism by which exposure to examples may lead to gains in principle knowledge.

This research focused on learning of an arithmetic principle as it applies in symbolic equations; however, the basic phenomenon may apply broadly, to other principles and other contexts. The use of comparison and contrast to highlight relevant similarities and differences has been shown to promote learning of different types of information (e.g., procedures, categories, conceptual knowledge) in several domains (Gentner & Medina, 1998; Hattikudur & Alibali, 2010; Kalish & Lawson, 2007; Namy & Clepper, 2010; Rittle-Johnson & Star, 2007). Comparing correct and incorrect examples may be particularly beneficial (Eryilmaz, 2002; Große & Renkl, 2007;

Huang et al., 2008; Siegler, 2002). The present work highlights the value of comparing correct and incorrect examples in the acquisition of principle knowledge.

Why Does Experience With Correct and Incorrect Examples Promote Learning?

We have suggested that experience with a mixture of correct and incorrect examples may promote learning by fostering improved encoding of the examples. In particular, comparing examples may help students to focus on key structural features of the problems, rather than more superficial features (e.g., Cummins, 1992; Gick & Paterson, 1992). Domain principles depend on structural features; for example, the Relation to Operands principle hinges on the relative magnitudes of the operands and result in equations. Thus, it makes sense that improved encoding of structural aspects of problems could lead to gains in principle knowledge.

In a similar vein, some recent work has suggested that perceptual learning is an integral component of learning to solve problems in mathematics and other domains. In studies of fraction and algebra learning, Kellman and colleagues (2008) used computer-based interventions designed to foster perceptual learning of problem structure—specifically, by providing students with practice in classifying problems in terms of their underlying structure. Students who received such opportunities for perceptual learning showed greater gains in problem-solving performance and transfer, relative to students who did not have such experience. These findings highlight the importance of fluency in encoding problem structure, particularly for generalization to novel contexts. This idea seems particularly applicable in the case of early arithmetic, which has been shown to have a strong perceptual component (Landy & Goldstone, 2007; McNeil & Alibali 2004).

Of course, it is unlikely that improvements in encoding or perceptual learning of problem structure are the only mechanisms by which experience with examples promotes learning. Other mechanisms are surely at play. One possibility is that experience with diverse examples may foster *abstraction of a general problem schema* (Catrambone & Holyoak, 1989). A problem schema abstracts away from the particulars of individual problems and retains key structural features. As such, abstraction of a problem schema could be one step in the process of building principle knowledge.

Another possibility is that experience with diverse examples, and in particular, experience with problems or solution strategies that are incorrect, may promote *deeper processing* of the material. This deeper processing may in turn make learners more likely to notice and encode regularities, some of which may tie to domain principles. This notion is similar to one proposed by Schwartz and Bransford (1998), who argued that experience with contrasting cases promotes the development of differentiated knowledge structures, which prepare the learner to understand subsequent material at a deeper level.

Limitations

The current experiments have several limitations that must be acknowledged. The manipulations in both experiments were relatively brief; each experiment was only about 20 minutes in duration in total, and in each experiment, the number of stimuli used in the training task was 40. Though low-knowledge participants showed changes in behavior even with a single, brief training session in both experiments, for effective use in educational settings, these ideas would need to be implemented during a longer period.

We showed in Experiment 2 that encoding practice fostered acquisition of principle knowledge; however, we did not provide direct evidence that exposure to diverse examples promotes improved encoding. The encoding measure involved only one equation, in part due to limitations on collecting data in the classroom setting. We believe that this brief, posttest-only measure may not have been sufficiently sensitive to detect differences in encoding in Experiment 1. It is also possible that some participants may have encoded relative magnitudes correctly but not used this encoded information to inform their principle understanding (see Siegler, 1989, for a discussion of encoded-but-not-used features), or used this encoded information in an incorrect way (i.e., they may have systematically chosen the violation as better than the nonviolation). Future work is needed to more directly address the role of problem encoding in principle knowledge acquisition. Experiments conducted in a laboratory setting would have more flexibility in assessing equation encoding.

Educational Implications

The present findings have implications for mathematics education. Continued and repetitive exposure to correct arithmetic equations may not be optimal for students' principle learning. This is not a completely novel idea; some educational practices include discussion of incorrect solutions or incorrect strategies (see Stigler & Hiebert, 1999). However, occasionally discussing an incorrect strategy produced by a student is quite different from deliberately structuring learning materials to include incorrect examples that may violate a domain principle.

Educators would likely also be interested in more multifaceted assessments of learners' principle knowledge. The current study, as with many studies of arithmetic principle knowledge, used a single-faceted knowledge assessment (see Prather & Alibali, 2009, for discussion of this issue). It would be beneficial both for applications to the classroom and for further explication of how arithmetic principle knowledge develops to expand this work to include different types of principle knowledge assessments and different types of problem formats (e.g., word problems or problems involving physical manipulatives).

Conclusion

In sum, the types of examples learners view, even if they do not work through those examples, can influence their learning of principles. Changes in how learners encode examples can also influence their principle learning. Although our data do not show this directly, we suggest that exposure to examples may lead to knowledge change by fostering improvements in encoding of problem features. Thus, a first step in building principle knowledge is learning what features deserve attention.

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APPENDIX

Mr. Jones's students are learning math.

Sometimes they get the wrong answers. The wrong answers are marked with an X. Take a look at the math these students did.

Tasha	Ruth
1) 12 − 4 = 8	1) 12 − 4 = 8
2) 10 − 2 = 5 X	2) 10 − 2 = 11 X
3) 13 − 1 = 9 X	3) 13 − 1 = 15 X
4) 14 − 4 = 5 X	4) 14 − 4 = 15 X
5) 5 − 1 = 4	5) 5 − 1 = 4

Who understands math better?
Tasha; Ruth; They are the same.